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Development of the Disturbing Function in Planetary Theory, in terms of the mean anomalies and constant elliptic elements.

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1. The following notation will be employed:—

a mean distance
f true anomaly
r radius vector
g mean anomaly
$\epsilon = \tan \frac{1}{2}\psi$ where $\sin \psi$ is the eccentricity.

Plain letters refer to the disturbed planet, accented letters to the disturbing planet.

ω, ω' the distance from the intersection of the orbits to the perihelia.

σ = sine of half the inclination of the two orbit planes.

2. Then the disturbing function (divided by the mass of the disturbing planet) is

$$R = \{r^2 + r'^2 - 2rr' \cos S\}^{-\frac{1}{2}} - \frac{r \cos S}{r'^2}$$

where $\cos S = \cos(-f + f' - \omega + \omega') - \sigma^2\{\cos(-f + f' - \omega + \omega') - \cos(f + f + \omega + \omega')\}$

If in this expression we replace r by a, r' by a', we have

$$R = \{a^2 + a'^2 - 2aa' \cos S\}^{-\frac{1}{2}} - \frac{a \cos S}{a'^2}$$

3. Suppose we have an expression

$$\phi(a)e^{\mathrm{I}f\sqrt{-1}}$$

and we require to replace a by r and f by its value in terms of g, and we wish to have the modified function expressed in a series of multiples of $e^{g\sqrt{-1}}$, we may adopt the following notation:

$$D = a\frac{d}{da},$$

$$\phi(r)e^{\mathrm{I}f\sqrt{-1}} =$$

$$\sum\sum \frac{\epsilon^n}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}\{n, j, \mathrm{I}, D\}e^{(\mathrm{I}+j)g\sqrt{-1}}$$

where the quantity in brackets $\{n, j, \mathrm{I}, D\}$ is an operator operating

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on $\phi(a)$ and the summation extends over all values of n, j that make the quantities under the factorial sign zero or positive integers.

Now suppose we have an expression

$$\phi(a,\ a')e^{(\mathrm{I}f+\mathrm{I}'f')\sqrt{-1}}$$

for the expansion of

$$\phi(r,\ r')e^{(\mathrm{I}f+\mathrm{I}'f')\sqrt{-1}}\ ;$$

we then obtain

$$\sum\sum\sum\frac{\epsilon^n\epsilon'^{n'}}{\left(\frac{n+j}{2}\right)!\ \left(\frac{n-j}{2}\right)!\ \left(\frac{n'+j'}{2}\right)!\ \left(\frac{n'-j''}{2}\right)!}\{n.j.\mathrm{I.D.}\}\{n'.j.\mathrm{I}'.\mathrm{D}'.\}e^{[(\mathrm{I}+j)g+(\mathrm{I}'+j')g']\surd}$$

the summation again extending over all values of $n, j,\ n', j'$ that make the quantities under the factorial signs zero or positive integers.

The advantage of employing the purely numerical denominator shown above is that the coefficients in the expanded values of $\{n, j,\ \mathrm{I},\ x\}$ are integers.

These operators are, moreover, rational integral algebraic functions of I, x of order n.

4. The values of $\{n, j,\ \mathrm{I},\ x\}$, as far as $n=6$, are given in the following table.

We have $\{n.\ -j.\ \mathrm{I}.\ x.\}=\{n.j.\ -\mathrm{I}.\ x\}$.

$\{0,\ 0,\ \mathrm{I},\ x\}=1$

$\{1,\ 1,\ \mathrm{I},\ x\}=\ +2\mathrm{I}\qquad -x$

$\{2,\ 2,\ \mathrm{I},\ x\}=\ +5\mathrm{I}+4\mathrm{I}^2-3x-4\mathrm{I}x+x^2$

$\{2,\ 0,\ \mathrm{I},\ x\}=\qquad -4\mathrm{I}^2+x\qquad +x^2$

5. These formulæ may be very easily developed from special cases. For instance, put $x=0$, $\mathrm{I}=1$, then we have

$$\cos\ (f+\theta)=\sum\sum\frac{\epsilon^n}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}\{n, j,\ 1,\ 0\}\ \cos\ (g+jg+\theta)$$

or

$$e^{f\sqrt{-1}}=\sum\sum\frac{\epsilon^n}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}\ \{n,\ j,\ 1,\ 0\}e^{(1+j)\,g\sqrt{-1}}$$

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Values of $\{n.\,j.\,\mathrm{I}.\,x\}$ *from* $n=3$ *to* $n=6$.

Power of x.	Power of I.	$\{3,3,\mathrm{I},x\}$	$\{3,1,\mathrm{I},x\}$	$\{4,4,\mathrm{I},x\}$	$\{4,2,\mathrm{I}.x\}$	$\{4,0,\mathrm{I},x\}$	$\{5,5,\mathrm{I},x\}$	$\{5,3,\mathrm{I},x\}$	$\{5,1,\mathrm{I},x\}$	$\{6,6,\mathrm{I},x\}$	$\{6,4,\mathrm{I},x\}$	$\{6,2,\mathrm{I},x\}$	$\{6,0,\mathrm{I},x\}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	+26	− 6	+206	−52	0	+2194	− 570	+ 70	+29352	− 7728	+1200	0
0	2	+30	−10	+283	−88	+23	+3360	−1008	+168	+48538	−14248	+2650	−280
0	3	+ 8	− 8	+120	−60	0	+1790	− 710	+170	+29835	−10600	+2513	0
0	4	...	...	+ 16	−16	+16	+ 400	− 240	+ 80	+ 8660	− 4060	+1300	−380
0	5	...	...	...	...	...	+ 32	− 32	+ 32	+ 1200	− 800	+ 400	0
0	6	...	...	...	...	...	...	...	...	+ 64	− 64	+ 64	− 64
1	0	−17	+ 5	−142	+40	− 6	−1569	+ 435	− 63	−21576	+ 5936	− 992	+ 60
1	1	−33	+ 5	−330	+71	0	−4080	+ 968	−100	−60752	+15054	−2136	0
1	2	−12	+ 4	−192	+48	0	−2995	+ 761	− 81	−51615	+13375	−1935	+ 39
1	3	...	...	− 32	+16	0	− 840	+ 296	− 24	−18860	+ 5920	− 964	0
1	4	...	...	...	...	...	− 80	+ 48	− 16	− 3120	+ 1360	− 304	− 48
1	5	...	...	...	...	...	...	...	...	− 192	+ 128	− 64	0
2	0	+ 9	+ 1	+ 95	7	− 9	+1220	− 176	− 26	+18694	− 3366	+ 46	+118
2	1	+ 6	+ 2	+102	+ 3	0	+1660	− 148	− 48	+29535	− 4320	− 211	0
2	2	...	...	+ 24	0	− 8	+ 660	− 60	− 36	+15345	− 2245	− 255	+195
2	3	...	...	...	...	...	+ 80	− 16	− 16	+ 3240	− 640	− 136	0
2	4	...	...	...	...	...	...	...	...	+ 240	− 80	− 16	+ 48
3	0	− 1	− 1	− 18	− 6	− 2	− 305	− 29	+ 13	− 5595	+ 105	+ 197	+ 57
3	1	...	...	− 8	− 4	0	− 230	− 42	+ 2	− 5530	− 180	+ 194	0
3	2	...	...	...	...	...	− 40	− 8	+ 8	− 1680	− 80	+ 112	+ 48
3	3	...	...	...	...	...	...	...	...	− 160	0	+ 32	0
4	0	...	...	+ 1	+ 1	+ 1	+ 30	+ 14	+ 6	+ 745	+ 165	+ 9	− 11
4	1	...	...	...	...	...	+ 10	+ 6	+ 2	+ 435	+ 130	+ 17	0
4	2	...	...	...	...	...	...	...	...	+ 60	+ 20	− 4	− 12
5	0	...	...	...	...	...	− 1	− 1	− 1	− 45	− 25	− 13	− 9
5	1	...	...	...	...	...	...	...	...	− 12	− 8	− 4	0
6	0	...	...	...	...	...	...	...	...	+ 1	+ 1	+ 1	+ 1

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where

$\{0 \quad 0 \quad 1 \quad 0\} = +\ 1$

$\{1 \quad 1 \quad 1 \quad 0\} = +\ 2$

$\{1 \quad -1 \quad 1 \quad 0\} = -\ 2$

$\{2 \quad 2 \quad 1 \quad 0\} = +\ 9$

$\{2 \quad 0 \quad 1 \quad 0\} = -\ 4$

$\{2 \quad -2 \quad 1 \quad 0\} = -\ 1$

$\{3 \quad 3 \quad 1 \quad 0\} = +\ 64$

$\{3 \quad 1 \quad 1 \quad 0\} = -\ 24$

$\{3 \quad -1 \quad 1 \quad 0\} = +\ 4$

$\{3 \quad -3 \quad 1 \quad 0\} = -\ 4$

$\{4 \quad 4 \quad 1 \quad 0\} = +625$

$\{4 \quad 2 \quad 1 \quad 0\} = -216$

$\{4 \quad 0 \quad 1 \quad 0\} = +\ 39$

$\{4 \quad -2 \quad 1 \quad 0\} = +\ 8$

$\{4 \quad -4 \quad 1 \quad 0\} = -\ 27$

$\{5 \quad 5 \quad 1 \quad 0\} = +\ 7776$

$\{5 \quad 3 \quad 1 \quad 0\} = -\ 2560$

$\{5 \quad 1 \quad 1 \quad 0\} = +\ 520$

$\{5 \quad -1 \quad 1 \quad 0\} = -\ 24$

$\{5 \quad -3 \quad 1 \quad 0\} = +\ 64$

$\{5 \quad -5 \quad 1 \quad 0\} = -\ 256$

$\{6 \quad 6 \quad 1 \quad 0\} = +117649$

$\{6 \quad 4 \quad 1 \quad 0\} = -\ 37500$

$\{6 \quad 2 \quad 1 \quad 0\} = +\ 8127$

$\{6 \quad 0 \quad 1 \quad 0\} = -\ 724$

$\{6 \quad -2 \quad 1 \quad 0\} = -\ 99$

$\{6 \quad -4 \quad 1 \quad 0\} = +\ 756$

$\{6 \quad -6 \quad 1 \quad 0\} = -\ 3125$

If this be regarded as a known elliptic formula, to be used as a starting-point, by raising to the I^th^ power we have

$$e^{\mathrm{I}f\sqrt{-1}} = \sum\sum \frac{\epsilon^n}{\left(\dfrac{n+j}{2}\right)!\left(\dfrac{n-j}{2}\right)!}\{n, j, \mathrm{I}, 0\}e^{(\mathrm{I}+j)g\sqrt{-1}}$$

thus verifying all the special cases of the table obtained by putting $x=0$.

6. Again, put $\mathrm{I}=0$ and we have

$$\phi(r) = \sum\sum \frac{\epsilon^n}{\left(\dfrac{n+j}{2}\right)!\left(\dfrac{n-j}{2}\right)!}\{n, j, 0.\ \mathrm{D}\}\phi(a)\cos jg$$

where

$\{0 \quad 0 \quad 0 \quad \mathrm{D}\} = 1$

$\{1 \quad 1 \quad 0 \quad \mathrm{D}\} = -\ \mathrm{D} = \{1 \quad -1 \quad 0 \quad \mathrm{D}\}$

$\{2 \quad 2 \quad 0 \quad \mathrm{D}\} = -\ 3\mathrm{D} + \mathrm{D}^2 = \{2 \quad -2 \quad 0 \quad \mathrm{D}\}$

$\{2 \quad 0 \quad 0 \quad \mathrm{D}\} = +\ \mathrm{D} + \mathrm{D}^2$

$\{3 \quad 3 \quad 0 \quad \mathrm{D}\} = -\ 17\mathrm{D} + 9\mathrm{D}^2 - \mathrm{D}^3 = \{3 \quad -3 \quad 0 \quad \mathrm{D}\}$

$\{3 \quad 1 \quad 0 \quad \mathrm{D}\} = +\ 5\mathrm{D} + \mathrm{D}^2 - \mathrm{D}^3 = \{3 \quad -1 \quad 0 \quad \mathrm{D}\}$

$\{4 \quad 4 \quad 0 \quad \mathrm{D}\} = -\ 142\mathrm{D} + 95\mathrm{D}^2 - 18\mathrm{D}^3 + \mathrm{D}^4 = \{4 \quad -4 \quad 0 \quad \mathrm{D}\}$

$\{4 \quad 2 \quad 0 \quad \mathrm{D}\} = +\ 40\mathrm{D} - 7\mathrm{D}^2 - 6\mathrm{D}^3 + \mathrm{D}^4 = \{4 \quad -2 \quad 0 \quad \mathrm{D}\}$

$\{4 \quad 0 \quad 0 \quad \mathrm{D}\} = -\ 6\mathrm{D} - 9\mathrm{D}^2 - 2\mathrm{D}^3 + \mathrm{D}^4$

$\{5 \quad 5 \quad 0 \quad \mathrm{D}\} = -\ 1569\mathrm{D} + 1220\mathrm{D}^2 - 305\mathrm{D}^3 + 30\mathrm{D}^4 - \mathrm{D}^5 = \{5 \quad -5 \quad 0 \quad \mathrm{D}\}$

$\{5 \quad 3 \quad 0 \quad \mathrm{D}\} = +\ 435\mathrm{D} - 176\mathrm{D}^2 - 29\mathrm{D}^3 + 14\mathrm{D}^4 - \mathrm{D}^5 = \{5 \quad -3 \quad 0 \quad \mathrm{D}\}$

$\{5 \quad 1 \quad 0 \quad \mathrm{D}\} = -\ 63\mathrm{D} - 26\mathrm{D}^2 + 13\mathrm{D}^3 + 6\mathrm{D}^4 - \mathrm{D}^5 = \{5 \quad -1 \quad 0 \quad \mathrm{D}\}$

$\{6 \quad 6 \quad 0 \quad \mathrm{D}\} = -21576\mathrm{D} + 18694\mathrm{D}^2 - 5595\mathrm{D}^3 + 745\mathrm{D}^4 - 45\mathrm{D}^5 + \mathrm{D}^6 = \{6 \quad -6 \quad 0 \quad \mathrm{D}\}$

$\{6 \quad 4 \quad 0 \quad \mathrm{D}\} = +\ 5936\mathrm{D} - 3366\mathrm{D}^2 + 105\mathrm{D}^3 + 165\mathrm{D}^4 - 25\mathrm{D}^5 + \mathrm{D}^6 = \{6 \quad -4 \quad 0 \quad \mathrm{D}\}$

$\{6 \quad 2 \quad 0 \quad \mathrm{D}\} = -\ 992\mathrm{D} + 46\mathrm{D}^2 + 197\mathrm{D}^3 + 9\mathrm{D}^4 - 13\mathrm{D}^5 + \mathrm{D}^6 = \{6 \quad -2 \quad 0 \quad \mathrm{D}\}$

$\{6 \quad 0 \quad 0 \quad \mathrm{D}\} = +\ 60\mathrm{D} + 118\mathrm{D}^2 + 57\mathrm{D}^3 - 11\mathrm{D}^4 - 9\mathrm{D}^5 + \mathrm{D}^6$

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but if $\log r = \log a + \rho$,

we know, by Maclaurin's Theorem, that

$$\phi(r) = \sum \frac{\rho^m}{m!} \mathrm{D}^m \phi(a).$$

Hence, in the above set of formulæ, picking out the coefficient of the first power of D, we have, on combining $\cos jg$ with $\cos -jg$,

$$\begin{array}{rlrl}
\rho = - & \epsilon & \cos g & \times 2 \\
- & \frac{1}{2}\epsilon^2 & 3 \cos 2g & \times 2 \\
+ & \epsilon^2 & & \\
- & \frac{1}{6}\epsilon^3 & 17 \cos 3g & \times 2 \\
+ & \frac{1}{2}\epsilon^3 & 5 \cos g & \times 2 \\
- & \frac{1}{24}\epsilon^4 & 142 \cos 4g & \times 2 \\
+ & \frac{1}{6}\epsilon^4 & 40 \cos 2g & \times 2 \\
- & \frac{1}{4}\epsilon^4\, 6 & & \\
- & \frac{1}{120}\epsilon^5 & 1569 \cos 5g & \times 2 \\
+ & \frac{1}{24}\epsilon^5 & 435 \cos 3g & \times 2 \\
- & \frac{1}{12}\epsilon^5 & 63 \cos g & \times 2 \\
- & \frac{1}{720}\epsilon^6 & 21576 \cos 6g & \times 2 \\
+ & \frac{1}{120}\epsilon^6 & 5936 \cos 4g & \times 2 \\
- & \frac{1}{48}\epsilon^6 & 992 \cos 2g & \times 2 \\
+ & \frac{1}{36}\epsilon^6\, 60 & &
\end{array}$$

If this is regarded as a known elliptic formula we may calculate

$$\frac{\rho^m}{m!} \text{ for } m = 2,\ 3,\ 4, \text{ etc.}$$

and thus verify the values of

$$\{n.\ j.\ 0.\ \mathrm{D}\}$$

7. Taking values of $\cos f$ and $\log \frac{r}{a}$ in terms of ϵ and g as known, we have now shown how to obtain

$$\{n, j, 0, \mathrm{D}\} \text{ and } \{n, j, \mathrm{I}, 0\}$$

We have lastly to obtain

$$\{n, j, \mathrm{I}, \mathrm{D}\}$$

We are supposed to start from $\phi(a)\, e^{\mathrm{I}f\sqrt{-1}}$ in which we

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require to replace a by r and f by its value in terms of g, and we then wish to pick out the coefficient of

$$\frac{\epsilon^n}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}e^{(1+j)g\sqrt{-1}}$$

We have seen how to replace $\phi(a)$ by a series whose typical term is

$$\frac{\epsilon^\nu}{\left(\frac{\nu+\gamma}{2}\right)!\left(\frac{\nu-\gamma}{2}\right)!}\{\nu.\gamma.0.\mathrm{D}\}e^{\gamma g\sqrt{-1}}$$

and $e^{\mathrm{I}f\sqrt{-1}}$ by a series whose typical term is

$$\frac{\epsilon^{n-\nu}}{\left(\frac{n-\nu+j-\gamma}{2}\right)!\left(\frac{n-\nu-j+\gamma}{2}\right)!}\{n-\nu,\ j-\gamma,\ \mathrm{I}.\ 0\}e^{(1+j-\gamma)g\sqrt{-1}}$$

Hence by multiplication we see that

$$,j,\mathrm{I.D}\} = \sum\sum\frac{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}{\left(\frac{\nu+\gamma}{2}\right)!\left(\frac{\nu-\gamma}{2}\right)!\left(\frac{n-\nu+j-\gamma}{2}\right)!\left(\frac{n-\nu-j+\gamma}{2}\right)!}\{\nu.\gamma.0.\mathrm{D}\}\{n-\nu.j-\gamma,\mathrm{I},$$

the summation extending over all values of ν, γ which make every quantity under the factorial sign in the denominator zero or a positive integer.

8. We have now merely to throw

$$\mathrm{R} = \{a^2+a'^2-2aa'\cos\mathrm{S}\}^{-\frac{1}{2}}-\frac{a\cos\mathrm{S}}{a'^2}$$

into the form

$$\sum\phi(a,\ a')\cos\{\mathrm{I}f+\mathrm{I}'f'+\theta\}$$

where θ, a combination of ω and ω', is a constant.

The second term of R is already in the required form, directly the value of cos S is substituted from § 2 of this paper.

We also have

$$\{a^2+a'^2-2aa'\cos\mathrm{S}\}^{-\frac{1}{2}}$$

$$=\sum_{k=0}^{k=\infty}\{a^2+a'^2-2aa'\cos(-f+f'-\omega+\omega')\}^{-k-\frac{1}{2}}(aa')^k\frac{(2k)!}{2^k(k!)^2}$$

$$\times\sigma^{2k}\{\cos(f+f'+\omega+\omega')-\cos(-f+f'-\omega+\omega')\}^k$$

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Now if a be the larger of the two quantities a, a', and if α be the ratio of the smaller to the larger,

$$\mathrm{a}\{a^2+a'^2-2aa'\cos\theta\}^{-k-\frac{1}{2}}(aa')^k \div \alpha^k$$

$$=\{1-\alpha e^{\theta\sqrt{-1}}\}^{-k-\frac{1}{2}}\{1-\alpha e^{-\theta\sqrt{-1}}\}^{-k-\frac{1}{2}}$$

$$=\left(1+\frac{k+\frac{1}{2}}{1}\alpha e^{\theta\sqrt{-1}}+\frac{(k+\frac{1}{2})\,.\,(k+\frac{3}{2})}{1\,.\,2}\alpha^2 e^{2\theta\sqrt{-1}}+\dots\right)$$

$$\times\left(1+\frac{k+\frac{1}{2}}{1}\alpha e^{-\theta\sqrt{-1}}+\frac{(k+\frac{1}{2})\,.\,(k+\frac{3}{2})}{1\,.\,2}\alpha^2 e^{-2\theta\sqrt{-1}}+\dots\right)$$

$$=1+\frac{(k+\frac{1}{2})^2}{1^2}\alpha^2+\frac{(k+\frac{1}{2})^2(k+\frac{3}{2})^2}{1^2\,.\,2^2}\alpha^4+\dots$$

$$+\frac{k+\frac{1}{2}}{1}\alpha\left(1+\frac{(k+\frac{1}{2}).(k+\frac{3}{2})}{1\,.\,2}\alpha^2+\frac{\{(k+\frac{1}{2})(k+\frac{3}{2})\}}{1\,.\,2}\,.\,\frac{\{(k+\frac{3}{2})(k+\frac{5}{2})\}}{2\,.\,3}\alpha^4+\dots\right)(\cos\theta+\cos$$

$$+\text{etc.}$$

$$=\sum_{i=\infty}^{i=\infty} b_{i.k}\cos i\,.\,\theta \quad \text{where} \quad b_{-i.k}=b_{i.k}$$

We now easily get

$$\mathrm{R}=\sum_{i=-\infty}^{i=\infty}\sum_{\kappa=0}^{\kappa=\infty}\sigma^{2\kappa}\mathrm{A}_{i.\kappa}.\cos\{i(-f+f'-\omega+\omega')+\kappa(f+f'+\omega+\omega')\}$$

where, sufficiently for a development as far as σ^6,

$$\mathrm{a}\mathrm{A}_{i.0}=\mathrm{a}\mathrm{A}_{-i.0}=b_{i.0}-\tfrac{1}{2}\sigma^2.\alpha\ \{b_{i-1.1}+b_{i+1.1}\}$$

$$+\tfrac{3}{8}\sigma^4\alpha^2\ \{b_{i-2.2}+4b_{i.2}+b_{i+2.2}\}$$

$$-\tfrac{5}{16}\sigma^6\alpha^3\{b_{i-3.3}+9b_{i-1.3}+9b_{i+1.3}+b_{i+3.3}\}$$

$$\mathrm{a}\mathrm{A}_{i.1}=\ \alpha\,b_{i.1}-\tfrac{3}{2}\sigma^2\alpha^2(b_{i-1.2}+b_{i+1.2})$$

$$+\tfrac{15}{8}\sigma^4\alpha^3(b_{i-2.3}+3b_{i.3}+b_{i+2.3})$$

$$\mathrm{a}\mathrm{A}_{i.2}=\tfrac{3}{4}\alpha^2 b_{i.2}-\tfrac{15}{8}\sigma^2\alpha^3(b_{i-1.3}+b_{i+1.3})$$

$$\mathrm{a}\mathrm{A}_{i.3}=\tfrac{5}{8}\alpha^3 b_{i.3}.$$

On account of the second term of R we must add

$$-\frac{1}{2}\frac{a}{a'^2}(1-\sigma^2) \quad \text{to} \quad \mathrm{A}_{1.0} \quad \text{and} \quad \mathrm{A}_{-1.0}$$

and

$$-\frac{a}{a'^2} \qquad \text{to} \quad \mathrm{A}_{0.1}.$$

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9. Summary :—

The value of the disturbing function can therefore be expressed in the perfectly general form

$$\sum_{i=-\infty}^{i=\infty}\sum_{k=0}^{k=\infty}\Sigma_n\Sigma_j\Sigma_{n'}\Sigma_{j'}$$

$$\frac{\epsilon^n.\epsilon'^{n'}}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!\left(\frac{n'+j'}{2}\right)!\left(\frac{n'-j'}{2}\right)!}\{n, j, \ -i+k, \ \mathrm{D}\}\{n', j', \ +i+k, \ \mathrm{D}'\}\mathrm{A}_{i.k}$$

$$\times\cos\{(-i+k+j)g+(i+k+j')g'+(-i+k)\omega+(i+k)\omega'\}$$

where since R is of the order -1 with respect to a, a' we may replace D′ by $-1-\mathrm{D}$ if we please, and where the value of every quantity required for an expansion as far as ϵ^6, ϵ'^6, σ^6 (including such combinations as $\epsilon^6\epsilon'^6\sigma^6$ of the eighteenth order) is given in this paper.

10. Numerical example :—

Required the coefficient of

$$\frac{\epsilon^3\epsilon'}{2!\,1!\,1!\,1!}\cos(-2g+4g'-3\omega+5\omega')$$

Here

$$i=4 \quad k=1 \quad n=3 \quad j=1 \quad n'=1 \quad j'=-1$$

The two operators are

$$\{3 \,.\, 1 \,.\, -3 \,.\, \mathrm{D}\}=+144+26\mathrm{D}-5\mathrm{D}^2-\mathrm{D}^3$$

$$\{1, \ -1, \ +5, \ \mathrm{D}'\}=\{1, \ +1, \ -5, \ \mathrm{D}'\}=-10-\mathrm{D}'=-9+\mathrm{D}$$

The quantity operated upon is $\sigma^2\mathrm{A}_{4.1}$

$$=\frac{\sigma^2}{\mathrm{a}}\left\{ab_{4.1}-\frac{3}{2}\sigma^2\alpha^2(b_{3.2}+b_{5.2})\right\}$$

stopping at $\sigma^4\epsilon^3\epsilon^1$ or the eighth order

$$b_{4.1}=\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{1\,.\,2\,.\,3\cdot4}\alpha^4\left\{1+\frac{\frac{11}{2}}{5}\,.\,\frac{\frac{3}{2}}{1}\alpha^2+\frac{\frac{11}{2}\cdot\frac{13}{2}}{5\,.\,6}\,\frac{\frac{3}{2}\cdot\frac{5}{2}}{1\,.\,2}\alpha^4+\,\ldots\ldots\right.$$

$$b_{3.2}=\frac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{1\,.\,2\,.\,3}\alpha^3\left\{1+\frac{\frac{11}{2}}{4}\,.\,\frac{\frac{5}{2}}{1}\alpha^2+\,\ldots\ldots\right.$$

$$b_{5.2}=\frac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1\,.\,2\,.\,3\cdot4\cdot5}\alpha^5\left\{1+\frac{\frac{15}{2}}{6}\,.\,\frac{\frac{5}{2}}{1}\alpha^2+\,\ldots\ldots\right.$$

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Further, if Venus be the disturbed planet and the Earth the disturbing planet, then

$$\begin{aligned} \alpha &= 0{\cdot}723\ 331\ 5 = a \\ \mathrm{a} &= 1. \qquad = a' \\ \log \sigma &= 8{\cdot}471\ 368 \\ \omega &= 234^\circ\ 7'\ 22''{\cdot}2 \qquad \epsilon = {\cdot}003\ 421\ 70 \\ \omega' &= 205^\circ\ 1'\ 29''{\cdot}1 \qquad \epsilon' = {\cdot}008\ 386\ 14 \end{aligned}$$

(see Newcomb, *A.P.*, iii. p. 406, 459)

and

$$\mathrm{D}\alpha^m = m\alpha^m .$$

The principal term becomes

$$-\frac{\epsilon^3\epsilon'}{2} \cdot 96 \cdot \sigma^2\alpha^5 \cdot \frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{1\cdot2\cdot3\cdot4}$$

which is less than 10^{-11}, and has still to be multiplied by the mass of the Earth.

Note on Major MacMahon's paper "On the Determination of the apparent Diameter of a fixed Star." By A. S. Eddington, M.A., M.Sc.

In Major MacMahon's paper (p. 126 *ante*), the fading of the light of a star at the instant of disappearance behind the Moon was treated purely as a problem of geometrical optics. When account is taken of the diffraction of the star's light by the limb of the Moon, his results will be found to require some modification. The argument given in this note appears to show that, judged by the rate of diminution of its light at occultation, every star will seem to have an angular diameter of about 0″·008, due to this diffraction.

Reducing the problem to its ideally simple form, we consider a point-star S. The Moon is treated as a screen M M′ with a straight edge through M perpendicular to the plane of the paper.

Let X E X′ be a screen in the position of the observer, E being the boundary of the geometrical shadow cast by the Moon. Let J be the intensity of illumination of the screen by the star when the Moon is out of the way, the true illumination of the screen will then be as follows:* —

Inside the geometrical shadow, the illumination is zero until near the edge; then it slowly increases, reaching the value $\frac{1}{4}$ J at E; outside the geometrical shadow it still increases, until it reaches

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[*] See, for instance, A. Schuster, *Theory of Optics*, p. 94.